Math 601 – Spring 2014 Final Exam

- **The exam is due on Thursday, May 8 at noon. Either email me the exam, or drop it off at my office or in my mailbox.**
- **You may use any references, including the notes and Hatcher. However, please cite any references that you use (including the page number if applicable).**
- **You may not discuss the problems on this exam with others until after the exam. In particular, do not work on the exam problems together, and do not request help with the problems from any online forum such as Math Stack Exchange.**

1. Explicitly describe a connected CW complex X with:
   - $\pi_1(X) \cong \langle \alpha, \beta \mid \alpha\beta\alpha = \beta\alpha\beta \rangle$
   - $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}/(3)$
   - $H_3(X) \cong \mathbb{Z}$
   - $H_n(X) \cong 0$ for all $n \geq 4$.

2. Recall that the Hopf fibration realizes S^3 as an S^1-bundle over S^2.
   (a) Provide a simple description for S^3 less the neighborhood of one fiber, and for S^3 less the neighborhood of two fibers.
   (b) Compute the fundamental group of S^3 less the neighborhood of k fibers.

3. Consider $f : S^n \longrightarrow S^n$ such that $f(x) = f(-x)$.
   (a) Show that the degree of f must be even.
   (b) Show that if n is even, then $\deg f = 0$
   (c) If n is odd, show that there exists such a map f with any even degree.

4. Let Y be an oriented, closed 3-manifold. Express $H_1(Y)$ as $F \oplus T$, where F is a free abelian group, and T is the torsion subgroup. Show that $H_2(Y) \cong F$.

5. Given two disjoint connected n-manifolds M_1 and M_2, their connected sum $M_1 \sharp M_2$ can be formed by deleting the interiors of closed n-balls $B_i \subset M_i$, and identifying ∂B_1 with ∂B_2 via some homeomorphism.
   (a) If M_1 and M_2 are closed and orientable n-manifolds, show that $H_i(M_1 \sharp M_2) \cong H_i(M_1) \oplus H_i(M_2)$ for $0 < i < n$.
   (b) Provide an example of closed manifolds M_1 and M_2 where $H_i(M_1 \sharp M_2) \ncong H_i(M_1) \oplus H_i(M_2)$

6. Is $S^2 \times S^2$ homotopy equivalent to $\mathbb{CP}^2 \sharp \mathbb{CP}^2$?

7. Let f be a homogeneous polynomial in $\mathbb{C}[x, y, z]$, so that the zero set of f is invariant under the $\mathbb{C}^*$ action. The zero set of f therefore induces a subset $V(f)$ of $\mathbb{CP}^2$. If ∇f is nonzero along the zero set of f, $V(f)$ is a submanifold of $\mathbb{CP}^2$ that is canonically oriented via the complex structure (via the inverse function theorem). For a given degree d, which homology classes does $[V(f)]$ realize?